Variational symmetries and Noether's theorem on time scales

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Outline of presentation

Motivation

- Emmy Noether showed in 1918 that for any variational symmetry of a problem of calculus of variations (with continuous time) there exists a conserved quantity along the respective Euler-Lagrange extremals.
- Calculus on time scales allows to study calculus of variations for arbitrary model of time (in particular, discrete time or mixed time).
- Our aim is to extend Noether's theorem to an arbitrary time scale.

Calculus on time scales

Origin

Calculus on time scales was developed by Stefan Hilger in 1988.
Reference: Bohner, M. and Peterson, A. (2001). *Dynamic Equations on Time Scales*, Birkhauser, Boston.

Definition

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers.

The *forward jump operator* is $\sigma : \mathbb{T} \to \mathbb{T}$

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The *backward jump operator* is $\rho : \mathbb{T} \to \mathbb{T}$

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The *graininess function* is $\mu : \mathbb{T} \to [0, \infty)$

$$\mu(t) := \sigma(t) - t.$$

Examples

If $\mathbb{T} = \mathbb{R}$ then for any $t \in \mathbb{R}$, $\sigma(t) = t = \rho(t)$ and $\mu(t) \equiv 0$.
If $\mathbb{T} = \mathbb{Z}$ then for every $t \in \mathbb{Z}$, $\sigma(t) = t + 1$, $\rho(t) = t - 1$ and $\mu(t) \equiv 1$.

Classification of points

A point $t \in \mathbb{T}$ is called:
right-scattered if $\sigma(t) > t$,
right-dense if $\sigma(t) = t$,
left-scattered if $\rho(t) < t$,
left-dense if $\rho(t) = t$,
isolated if it is both left-scattered and right-scattered,
dense if it is both left-dense and right-dense.

Truncated time scale

Let

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Definition of delta derivative

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. *Delta derivative* of f at t is the real number $f^\triangle(t)$ with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ (for some $\delta > 0$) such that

$$|(f(\sigma(t)) - f(s)) - f^\triangle(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. We say that f is *delta differentiable* on $\mathbb{T}$ provided $f^\triangle(t)$ exists for all $t \in \mathbb{T}$.

Remark

If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff f is differentiable in the ordinary sense at t. Then $f^\triangle(t) = f'(t)$.
If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \to \mathbb{R}$ is always delta differentiable at every $t \in \mathbb{Z}$ with $f^\triangle(t) = f(t + 1) - f(t)$.

Definition

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at the right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.
The set of all rd-continuous functions is denoted by $\mathcal{C}_{rd}$.

Definition

A function $F : \mathbb{T} \to \mathbb{R}$ is called a *antiderivative of* $f : \mathbb{T} \to \mathbb{R}$ if it satisfies $F^{\Delta}(t) = f(t)$, for all $t \in \mathbb{T}^{\kappa}$.
Then the *indefinite integral* of f is defined by $\int f(t)\Delta t = F(t) + C$, where C is arbitrary constant. The *Cauchy integral* of f is defined by $\int_r^s f(t)\Delta t = F(s) - F(r)$, for all $s, t \in \mathbb{T}$.

It is known that every rd-continuous function has an antiderivative.

Example

If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the usual Riemann integral.

If $\mathbb{T} = h\mathbb{Z}$, where $h > 0$, then

$$\int_a^b f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} h \cdot f(kh),$$

for $a < b$.

Invariance without time transformation

Optimization problem

Let $\mathbb{T}$ be an arbitrary time scale, $a, b \in \mathbb{T}$ and $q : [a, b] \to \mathbb{R}^n$ be delta differentiable, where $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$.
Consider the optimization problem

$$I[q(\cdot)] = \int_a^b L\left(t, q^\sigma(t), q^\Delta(t)\right) \Delta t \longrightarrow \min$$

under the boundary conditions $q(a) = q_a$ and $q(b) = q_b$.
The Lagrangian $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is assumed to be a C^1-function with respect to all its arguments.

Definition 1

Let U be a set of delta differentiable functions $q : [a, b] \to \mathbb{R}^n$. Functional I is said to be *invariant* on U under a one-parameter family of state transformations

$$\bar{q} = q + \varepsilon\xi(t, q) + o(\varepsilon) \tag{1}$$

if, and only if,

$$\int_{t_a}^{t_b} L\left(t, q^\sigma(t), q^\Delta(t)\right)\Delta t = \int_{t_a}^{t_b} L\left(t, \bar{q}^\sigma(t), \bar{q}^\Delta(t)\right)\Delta t \tag{2}$$

for any subinterval $[t_a, t_b] \subseteq [a, b]$ with $t_a, t_b \in \mathbb{T}$, for any ε and for any $q \in U$, where $\bar{q}(t) = q(t) + \varepsilon\xi(t, q(t)) + o(\varepsilon)$.

Let us denote by $\partial_i L$ the partial derivative of L with respect to its i-th argument, for $i \geq 1$.

Theorem

If functional I is invariant on U under transformations (1), then

$$\begin{aligned} \partial_2 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi^\sigma(t, q(t)) + \\ \partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi^\Delta(t, q(t)) = 0 \end{aligned} \tag{3}$$

for all $t \in [a, b]$ and all $q \in U$, where
$\xi^\sigma(t, q(t)) = \xi(\sigma(t), q(\sigma(t)))$ and $\xi^\Delta(t, q(t)) = \frac{\Delta}{\Delta t}\xi(t, q(t))$.

Proof

Having in mind that condition

$$\int_{t_a}^{t_b} L\left(t, q^\sigma(t), q^\Delta(t)\right) \Delta t = \int_{t_a}^{t_b} L\left(t, \bar{q}^\sigma(t), \bar{q}^\Delta(t)\right) \Delta t$$

is valid for any subinterval $[t_a, t_b] \subseteq [a, b]$, we can rid off the integral signs in it. Thus it is equivalent to

$$L\left(t, q^\sigma(t), q^\Delta(t)\right) = \\ L(t, q^\sigma(t) + \varepsilon\xi^\sigma(t, q(t)) + o(\varepsilon), q^\Delta(t) + \varepsilon\xi^\Delta(t, q(t)))\,. \quad (4)$$

Differentiating both sides of equation (4) with respect to ε, then setting $\varepsilon = 0$, we obtain equality (3).

Definition

Quantity $C(t, q, q^\sigma, q^\Delta)$ is said to be a *conservation law* for functional I on U if, and only if,

$$\frac{\Delta}{\Delta t} C(t, q(t), q^\sigma(t), q^\Delta(t)) = 0$$

along all $q \in U$ that satisfy the Euler-Lagrange equation

$$\frac{\Delta}{\Delta t} \partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right) = \partial_2 L\left(t, q^\sigma(t), q^\Delta(t)\right) . \qquad (5)$$

Theorem 1

If functional I is invariant on U under the one-parameter family of transformations (1), then

$$C(t, q, q^\sigma, q^\Delta) = \partial_3 L\left(t, q^\sigma, q^\Delta\right) \cdot \xi(t, q) \tag{6}$$

is a conservation law.

Proof

Using the Euler-Lagrange equation (5) and the necessary condition of invariance (3), we obtain:

$$\begin{aligned}
&\frac{\Delta}{\Delta t}\left(\partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi(t, q(t))\right) \\
&\quad = \frac{\Delta}{\Delta t}\partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi^\sigma(t, q(t)) \\
&\quad + \partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi^\Delta(t, q(t)) \\
&\quad = \partial_2 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi^\sigma(t, q(t)) \\
&\quad + \partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi^\Delta(t, q(t)) \\
&\quad = 0\,.
\end{aligned}$$

Invariance with time transformation

Let us consider now the one-parameter family of transformations

$$\begin{cases} \bar{t} = T_\varepsilon(t,q) = t + \varepsilon\tau(t,q) + o(\varepsilon)\,, \\ \bar{q} = Q_\varepsilon(t,q) = q + \varepsilon\xi(t,q) + o(\varepsilon)\,. \end{cases} \tag{7}$$

Let as before U be a set of delta differentiable functions $q : [a,b] \to \mathbb{R}^n$.
We assume that for every $q \in U$ and every ε, the map $[a,b] \ni t \mapsto \alpha(t) := T_\varepsilon(t, q(t)) \in \mathbb{R}$ is increasing and its image is again a time scale with the forward shift operator $\bar{\sigma}$ and the delta derivative $\bar{\Delta}$. Observe that the following holds: $\bar{\sigma} \circ \alpha = \alpha \circ \sigma$.
Let $\beta = \alpha^{-1}$. We set $\bar{q}(\bar{t}) := Q\left(\beta(\bar{t}), q(\beta(\bar{t}))\right)$.

Definition 2

Functional I is said to be invariant on U under the group of infinitesimal transformations (7) if, and only if, for any subinterval $[t_a, t_b] \subseteq [a, b]$, any ε and any $q \in U$

$$\int_{t_a}^{t_b} L\left(t, q^\sigma(t), q^\Delta(t)\right) \Delta t = \int_{T(t_a,q(t_a))}^{T(t_b,q(t_b))} L\left(\bar{t}, \bar{q}^{\bar{\sigma}}(\bar{t}), \bar{q}^{\bar{\Delta}}(\bar{t})\right) \bar{\Delta}\bar{t}.$$

The main result

Theorem

If functional I is invariant on U under the group of infinitesimal transformations (7), then

$$C(t,q,q^\sigma,q^\Delta) = \partial_3 L\left(t,q^\sigma,q^\Delta\right)\cdot\xi(t,q) + \left[L(t,q^\sigma,q^\Delta) - \partial_3 L\left(t,q^\sigma,q^\Delta\right)\cdot q^\Delta - \partial_1 L\left(t,q^\sigma,q^\Delta\right)\cdot\mu(t)\right]\cdot\tau(t,q) \tag{8}$$

is a conservation law.

Example

Let $\mathbb{T} = \{2^n : n \in \mathbb{N} \cup \{0\}\}$ and

$$L(t, q^\sigma, q^\Delta) = \frac{(q^\sigma)^2}{t} + t(q^\Delta)^2$$

for $q \in \mathbb{R}$. It can be shown that the functional I is invariant under the family of transformations:

$$\bar{t} = te^\varepsilon = t + t\varepsilon + o(\varepsilon), \;\; \bar{q} = q.$$

Then, Noether's theorem generates the following conservation law:

$$C(t, q^\sigma, q^\Delta) = 2\left[\frac{(q^\sigma)^2}{t} - t(q^\Delta)^2\right] \cdot t.$$

Corollary: Classical result

Classical Noether's theorem

Let $\mathbb{T} = \mathbb{R}$. If functional I is invariant on U under the group of infinitesimal transformations (7), then

$$C(t, q, q') = \partial_3 L\left(t, q, q'\right) \cdot \xi(t, q) + \left[L(t, q, q') - \partial_3 L\left(t, q, q'\right) \cdot q'\right] \cdot \tau(t, q)$$

is a conservation law.

Corollary: discrete time

The result for the discrete time is new

Let $\mathbb{T} = \mathbb{Z}$. If functional I is invariant on U under the group of infinitesimal transformations (7), then

$$\begin{aligned} C(t, q, q^+, \Delta q) = \partial_3 L\left(t, q^+, \Delta q\right) \cdot \xi(t, q) + \\ \left[L(t, q^+, \Delta q) - \partial_3 L\left(t, q^+, \Delta q\right) \cdot \Delta q \right. \\ \left. - \partial_1 L\left(t, q^+, \Delta q\right)\right] \cdot \tau(t, q) \end{aligned}$$

is a conservation law, where $q^+(t) = q(t+1)$ and $\Delta q = q^+ - q$.

Sketch of the proof of the main result

We will show that invariance of I under (7) (in the sense of Definition 2) is equivalent to invariance of another functional $\tilde{I}$ in the sense of Definition 1.

Let $\tilde{L}(t; s, q; r, v) := L(s - \mu(t)r, q, \frac{v}{r}) \cdot r$ for $q, v \in \mathbb{R}^n$, $t \in [a, b]$ and $s, r \in \mathbb{R}$, $r \neq 0$. $\tilde{L}$ is a new Lagrangian with the state variable $(s, q) \in \mathbb{R}^{n+1}$. Observe that for $s(t) = t$ and any $q : [a, b] \to \mathbb{R}^n$

$$L(t, q^\sigma(t), q^\Delta(t)) = \tilde{L}(t; s^\sigma(t), q^\sigma(t); s^\Delta(t), q^\Delta(t))$$

so for the functional

$$\tilde{I}[s(\cdot), q(\cdot)] := \int_a^b \tilde{L}(t; s^\sigma(t), q^\sigma(t); s^\Delta(t), q^\Delta(t))\Delta t$$

we get $I[q(\cdot)] = \tilde{I}[s(\cdot), q(\cdot)]$ whenever $s(t) = t$.

Consider the family of transformations $(T_\varepsilon, Q_\varepsilon)$ given by (7) and let $q \in U$. From the invariance of I, for $s(t) = t$, we get

$$\tilde{I}[s(\cdot), q(\cdot)] = I\left[q(\cdot)\right] = \tilde{I}[\alpha(\cdot), (\bar{q} \circ \alpha)(\cdot)].$$

This implies that $\tilde{I}$ is invariant on $\tilde{U} = \{(s, q) \mid s(t) = t,\ q \in U\}$ under the group of state transformations

$$(\bar{s}, \bar{q}) = (T_\varepsilon(s, q), Q_\varepsilon(s, q))$$

in the sense of Definition 1.
Applying Theorem 1, we obtain that for $s(t) = t$

$$\begin{aligned} C\left(t, s, q, s^\sigma, q^\sigma, s^\Delta, q^\Delta\right) &= \partial_5 \tilde{L}(t; s^\sigma, q^\sigma; s^\Delta, q^\Delta) \cdot \xi(s, q) \\ &\quad + \partial_4 \tilde{L}(t; s^\sigma, q^\sigma; s^\Delta, q^\Delta) \cdot \tau(s, q) \end{aligned} \quad (9)$$

is a conservation law.

For $s(t) = t$ we can compute

$$\partial_5 \bar{L}(t; s^\sigma, q^\sigma; s^\Delta, q^\Delta) = \partial_3 L\left(t, q^\sigma, q^\Delta\right) \tag{10}$$

and

$$\begin{aligned} &\partial_4 \bar{L}(t; s^\sigma, q^\sigma; s^\Delta, q^\Delta) = \\ &\qquad L(t, q^\sigma, q^\Delta) - \partial_3 L\left(t, q^\sigma, q^\Delta\right) \cdot q^\Delta - \partial_1 L\left(t, q^\sigma, q^\Delta\right) \cdot \mu(t). \end{aligned} \tag{11}$$

Substituting (10) and (11) into (9) we arrive to the intended conclusion (8).

Reference

Z. Bartosiewicz and D. F. M. Torres, Noether's theorem on time scales, *Journal of Mathematical Analysis and Applications* 342 (2008)