LYAPUNOV THEOREMS ON ASYMPTOTIC STABILITY OF DYNAMIC SYSTEMS ON TIME SCALES

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Outline of presentation

Introduction

- The theory of time scales allows to unify the seemingly disparate fields of discrete and continuous dynamical systems.
- The concept of stability is extremely important, because almost every workable system is designed to be stable.
- The Lyapunov's direct method uses a generalized energy function to study the stability of the solutions.
- The stability of dynamic systems can be obtained without any prior knowledge of the solutions.
- Examining stability of dynamic systems on time scales not only unifies the two special cases of continuous and discrete time, but it also extends these notions for arbitrary time scales.

The concept of time scales

Definition of time scales

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

Examples of time scales:

$\mathbb{R}$, $h\mathbb{Z}$, $h > 0$, $\mathbb{N}$, $\mathbb{N}_0$, $[0,1] \cup [2,3]$, $2^{\mathbb{N}_0}$,

$P_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$ or Cantor set.

Remark:

The following sets $\mathbb{Q}$, $\mathbb{C}$ or $(0,1)$ are not time scales.

Let $t \in \mathbb{T}$.

Definition: The forward jump operator

is function $\sigma : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$.

Definition: *The backward jump operator*

is function $\rho : \mathbb{T} \to \mathbb{T}$ defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$

Definition: *The graininess function*

is function $\mu : \mathbb{T} \to [0, \infty)$ defined by $\mu(t) := \sigma(t) - t$.

Classification of points: A point $t \in \mathbb{T}$ is called

- *right-scattered* if $\sigma(t) > t$ and *right-dense* if $\sigma(t) = t$
- *left-scattered* if $\rho(t) < t$ and *left-dense* if $\rho(t) = t$.
- *isolated* if it is both left-scattered and right-scattered
- *dense* if it is both left-dense and right-dense.

Definition of delta derivative

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Define $f^\Delta(t)$ as the number (when it exists), with the property that, for any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \ \forall s \in U.$$

The function $f^\Delta : \mathbb{T} \to \mathbb{R}$ is called the *delta derivative* of f on $\mathbb{T}^k$. f is *delta differentiable* on $\mathbb{T}^k$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$.

Definition of nabla derivative

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$. Define $f^\nabla(t)$ as the number (when it exists), with the property that, for any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|, \ \forall s \in U.$$

Let $\mathbb{T}$ be a time scale, $a < b$ be points in $\mathbb{T}$ and $(a, b)_\square$ be a half-closed bounded interval in $\mathbb{T}$, where $\square$ denotes either Δ or ∇ operators and $(a, b)_\Delta := [a, b)$, $(a, b)_\nabla := (a, b]$.
Let $P = \{t_0, t_1, \ldots, t_n\} \subset [a, b]$, where $a = t_0 < t_1 < \ldots < t_n = b$ be a partition of $(a, b)_\square$ and f be a real-valued bounded function on $(a, b)_\square$.

Lemma

For every $\delta > 0$ there exists at least one partition $P: \ a = t_0 < t_1 < \ldots < t_n = b$ of $(a, b)_\square$ such that for each $i \in 1, 2, \ldots, n$ either $t_i - t_{i-1} \leq \delta$ or $t_i - t_{i-1} > \delta$ and $\rho(t_i) = t_{i-1}$ for $\square = \Delta$ or $\sigma(t_{i-1}) = t_i$ for $\square = \nabla$.

For given $\delta > 0$ we denote by $\mathfrak{P}_{\delta\square}$ the set of all partitions $P_\square: \ a = t_0 < t_1 < \ldots < t_n = b$ that possess the properties indicated in above lemma.

Let f be bounded function on $(a,b)_\square$, and let $P: \ a = t_0 < t_1 < \ldots < t_n = b$ be a partition of $(a,b)_\square$. In each interval $(t_{i-1}, t_i)_\square$, where $1 \le i \le n$, let us choose an arbitrary point ξ_i and form the Riemann $\square$-sum of f corresponding to the partition P

$$S_\square = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}).$$

We say that f is Riemann $\square$-integrable from a to b if there exists a number $I_\square$ with the following property. For each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S_\square - I_\square| < \varepsilon$ for every Riemann $\square$-sum $S_\square$ of f corresponding to a partition $P \in \mathfrak{P}_{\delta\square}$ independent of the way in which we choose $\xi_i \in (t_{i-1}, t_i)_\square$, $i = 1, 2, \ldots, n$. It is easily seen that such number $I_\square$ is unique. The number $I_\square$ is called the Riemann $\square$-integral of f from a to b.

Proposition

Let $a, b \in \mathbb{T}$ and $f : [a, b] \to \mathbb{R}$ be strictly decreasing. Then

$$\int_a^b f(t)\nabla t \leq \int_a^b f(t)dt \leq \int_a^b f(t)\Delta t.$$

Proposition

Let $a, b \in \mathbb{T}$ and $f : [a, b] \to \mathbb{R}$ be strictly increasing. Then

$$\int_a^b f(t)\Delta t \leq \int_a^b f(t)dt \leq \int_a^b f(t)\nabla t.$$

Theorem: Chain Rule for ∇-derivative

Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly decreasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $g : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $g^{\widetilde{\nabla}}(\nu(t))$ and $\nu^{\Delta}(t)$ exist for $t \in \mathbb{T}^k$, then

$$\frac{\nabla}{\nabla t} g(\nu(t)) = g^{\widetilde{\Delta}}(\nu(t))\nu^{\nabla}(t).$$

Corollary

Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly decreasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Then

$$\frac{1}{\nu^{\nabla}(t)} = (\nu^{-1})^{\widetilde{\Delta}}(\nu(t))$$

at points where $\nu^{\nabla}(t)$is different from zero.

Theorem: Substitution rule for Δ-integral

Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly decreasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $f : \mathbb{T} \to \mathbb{R}$ is *rd*-continuous function and ν is Δ-differentiable with *rd*-continuous derivative, then for $a, b \in \mathbb{R}$

$$\int_a^b f(t)\nu^{\Delta}(t)\Delta t = -\int_{\nu(b)}^{\nu(a)} f(\nu^{-1}(s))\widetilde{\nabla} s.$$

Corollary

Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly decreasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $g : \widetilde{\mathbb{T}} \to \mathbb{R}$ is *rd*-continuous function and ν is Δ-differentiable with *rd*-continuous derivative, then for $a, b \in \mathbb{R}$

$$\int_a^b g(\nu(t))\nu^{\Delta}(t)\Delta t = -\int_{\nu(b)}^{\nu(a)} g(s)\widetilde{\nabla} s.$$

We shall assume that $\sup \mathbb{T} = +\infty$. Consider a Δ-differential equation on time scale

$$x^{\Delta} = f(x, t), \tag{1}$$

where $x = x(t) \in \mathbb{R}^n$ and $t \in \mathbb{T}$. We shall assume that $x = 0$ is an equilibrium of (1), i.e. the function $x(t) = 0$ for $t \in \mathbb{T}$ satisfies (1). This means that $f(0, t) = 0$ for all $t \in \mathbb{T}$. By a forward trajectory determined by $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{T}$ we shall mean a function $x : [t_0, +\infty) \cap \mathbb{T} \to \mathbb{R}^n$ that satisfies (1) for all $t \geq t_0$ and satisfies the initial condition $x(t_0) = x_0$. A backward trajectory is defined in a similar way as a function $x : (\inf \mathbb{T}, t_0] \cap \mathbb{T} \to \mathbb{R}^n$. We shall assume that there is $h > 0$ such that for any $t_0 \in \mathbb{T}$ and any x_0 from the ball $K_h := \{x \in \mathbb{R}^n \ : \ |x| < h\}$ there exists a unique forward trajectory of (1) staying in K_h. The forward trajectory determined by x_0 and t_0, and evaluated at the time t is denoted by $p(t, x_0, t_0)$.

The following example shows that we have uniqueness of forward trajectories and lack of uniqueness of backward trajectories.

Example

Let $\mathbb{T} = \mathbb{R}$. Consider the equation $\dot{x} = -sgn\ x\sqrt{|x|}$, with the initial condition $x(0) = x_0$. Then the unique forward trajectory $x(t) = \frac{1}{4}(t - 2\sqrt{|x_0|})^2$ reaches the equilibrium 0 when $t \to 2\sqrt{|x_0|}$. This means that there are infinitely many backward trajectories starting from 0 at any fixed time.

Remark

A behavior that appears in above example is quite common for discrete time scales. Since $\mu(t)$ is then always positive, (1) may be rewritten as

$$x(t+\mu(t)) = x(t) + \mu(t)f(x(t), t).$$

If for some t_1 and for some $x(t_1)$, $x(t_1) + \mu(t_1)f(x(t_1), t_1) = 0$ then the solution becomes 0 in finite time. As before, this means that backward trajectories are not unique. On the other hand, forward trajectories always exist and are unique. This lack of symmetry follows from the fact that we use the delta derivative in the description of the system and not the nabla derivative.

Definition of function class $\mathcal{K}$

A function $\varphi : [0, h] \to [0, +\infty)$ belongs to class $\mathcal{K}$ ($\varphi \in \mathcal{K}$) if it is continuous and increasing, and if $\varphi(0) = 0$.

Definition of function class $\mathcal{L}$

A function $\zeta : [0, +\infty) \to (0, +\infty)$ belongs to class $\mathcal{L}$ ($\zeta \in \mathcal{L}$) if it is continuous, decreasing on $[t_0, +\infty)$ for some $t_0 \geq 0$, and if $\lim_{s \to \infty} \zeta(s) = 0$.

Definition of decrescent function

The function $\upsilon : K_h \times \mathbb{T} \to \mathbb{R}$ is called decrescent, if there exists a function φ of class K and $t_0 \in \mathbb{T}$ such that for all $t \geq t_0$ and $x \in K_h$

$$\upsilon(x, t) \leq \varphi(\|x\|).$$

Definition of positive definite function

A function $\upsilon : K_h \times \mathbb{T} \to \mathbb{R}$ is called positive definite if there exists a function $\varphi(r)$ of the class K and $t_0 \in \mathbb{T}$ such that for all $t \geq t_0$ and $x \in K_h$

$$\upsilon(x, t) \geq \varphi(\|x\|).$$

Definition of negative definite function

The function υ is called negative definite if $-\upsilon$ is positive definite.

Let $\upsilon : K_h \times \mathbb{T} \to \mathbb{R}$ and let $a \in K_h$ and $t_0 \in \mathbb{T}$. Consider the forward trajectory $p(\cdot, a, t_0)$ and the composition

$$\widetilde{\upsilon}(t) := \upsilon(p(t, a, t_0), t)$$

Then the delta derivative $\widetilde{\upsilon}^{\Delta}(t_0)$ is called the delta derivative of function υ with respect to the system (1) at (a, t_0) and denoted by $\upsilon^{\Delta}(a, t_0)$.

Definition of uniform stability

The equilibrium of (1) is called uniformly stable if there exists a comparison function $\varphi(r) \in \mathrm{K}$ such that for all $t_0 \in \mathbb{T}$, $a \in K_h$ and $t \geq t_0$

$$\|p(t, a, t_0)\| \leq \varphi(\|a\|).$$

Definition of uniform asymptotic stability

The equilibrium is called uniformly asymptotically stable if there exist $\varphi \in \mathrm{K}$ and $\zeta \in \mathrm{L}$ such that for all $t_0 \in \mathbb{T}$, $a \in K_h$ and $t \geq t_0$

$$\|p(t, a, t_0)\| \leq \varphi(\|a\|)\zeta(t - t_0).$$

The following example shows that there are time scales for which no system can be stable.

Example

Consider equation

$$x^{\Delta} = ax, \quad a \in \mathbb{C} \tag{2}$$

on time scale $\mathbb{T} = 2^{\mathbb{N}_0}$. Let $k_0 \in \mathbb{N}_0$ be such that for all $k \geqslant k_0$, $|1 + a2^k| \geqslant 1$. Then for $t_0 = 2^{k_0} \in \mathbb{T}$ the solution of (2) on that time scale is equal

$$x(2^k) = (1 + 2^{k-1}a)(1 + 2^{k-2}a)\dots(1 + 2^{k_0}a)x(2^{k_0}).$$

For $x(2^{k_0}) \neq 0$ it does not converge to zero when k goes to infinity. Hence (2) is not stable.

The following lemma is an extension of the inequality presented in [W. Hahn, *Stability of Motion*].

Lemma

Let us consider the equation $y^{\Delta} = -g(y)$, where $y \in [0, h]$ and $g \in K$. Assume that for every $t_0 \in \mathbb{T}$ and every initial condition $y(t_0) = y_0 \in [0, h]$, there exists a global forward solution y, such that $y(t) \in [0, h]$. Then there are $\varsigma \in \mathrm{K}$ and $\zeta \in \mathrm{L}$ such that for every $t_0 \in \mathbb{T}$, every $y_0 \in [0, h]$ and every $t \geq t_0$, $t \in \mathbb{T}$, the corresponding solution satisfies

$$y(t) \leq \varsigma(y_0)\zeta(t - t_0). \tag{3}$$

In particular, if $t \to +\infty$, then $y(t) \to 0$.

The extensions of Lyapunov theorems to arbitrary time scales.

Theorem on uniform stability

If there exists a positive definite, decrescent function $\upsilon : K_h \times \mathbb{T} \to \mathbb{R}$ with a non-positive delta derivative with respect to (1), then the equilibrium of (1) is uniformly stable.

Theorem on uniform asymptotic stability

If there exists a positive definite, decrescent function $\upsilon : K_h \times \mathbb{T} \to \mathbb{R}$ with negative definite derivative with respect to (1) then the equilibrium is uniformly asymptotically stable.

The following lemma is an extension of Massera lemma on arbitrary time scales.

Lemma

Let $\beta \in \mathtt{L}$ and let ξ be a continuous, positive and non-decreasing function on $[0, +\infty)$. Then there exists a function α of class $\mathtt{K}$ defined on $[0, \beta(0)]$, such that for any positive continuous function $\beta^* : [0, +\infty) \to \mathbb{R}$ that satisfies $\beta^*(\tau) \leq \beta(\tau)$ for all $\tau \geq 0$, the integral

$$\int_0^\infty \alpha(\beta^*(\tau))\xi(\tau)\Delta\tau$$

is convergent on an arbitrary time scale $\mathbb{T}$.

Converse theorem on uniform stability

Let us assume that the equilibrium for (1) is uniformly stable. Then there exists a positive definite decrescent function $\omega : K_h \times \mathbb{T} \to [0, +\infty)$ that is non-increasing along forward trajectories.

Converse theorem on uniform asymptotic stability

Let us assume that the equilibrium for (1) be uniformly asymptotically stable. Then there exists a positive definite decrescent function $\upsilon : K_h \times \mathbb{T} \to [0, +\infty)$ such that for any forward trajectory, either υ is decreasing along this trajectory or it is decreasing until some $t_1 \in \mathbb{T}$ and is 0 for $t \geq t_1$.

Conclusions

- The direct Lyapunov's method based on scalar auxiliary function proves to be a powerful technique of qualitative analysis of dynamic systems.
- Lyapunov functions are a type of distance and their delta derivative along the forward trajectory describes how the distance varies in time.
- By using scalar Lyapunov functions we get necessary and sufficient conditions for asymptotic stability of time-variant systems on arbitrary time scales.

References

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