Weak Observability of Small Solutions of Linear Differential-Algebraic Systems with one Delay

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Definition of linear differential-algebraic systems with delays (DAD)

In this presentation we study the simplest linear time invariant differential-algebraic systems with delays (DAD):

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t), \ \ t > 0, \tag{1a}$$

$$x_2(t) = A_{21}x_1(t) + A_{22}x_2(t-h), \ \ t \geq 0, \tag{1b}$$

with output $$z(t) = B_1x_1(t) + B_2x_2(t), \tag{1c}$$

where $x_1(t) \in \mathbb{R}^{n_1}, \ x_2(t) \in \mathbb{R}^{n_2}, \ z(t) \in \mathbb{R}^r, t \geq 0$; the matrices are real of proper dimensions. System (1) should be completed with initial conditions:

$$x_1(+0) = x_0, \ x_2(\tau) = \psi(\tau), \ \tau \in [-h, 0], \tag{2}$$

where $x_0 \in \mathbb{R}^{n_1}$; $\psi \in PC([-h, 0], \mathbb{R}^{n_2})$ and $PC([-h, 0], \mathbb{R}^{n_2})$ denotes the set of piecewise continuous m-vector-functions in $[-h, 0]$.

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Small solutions of DAD systems

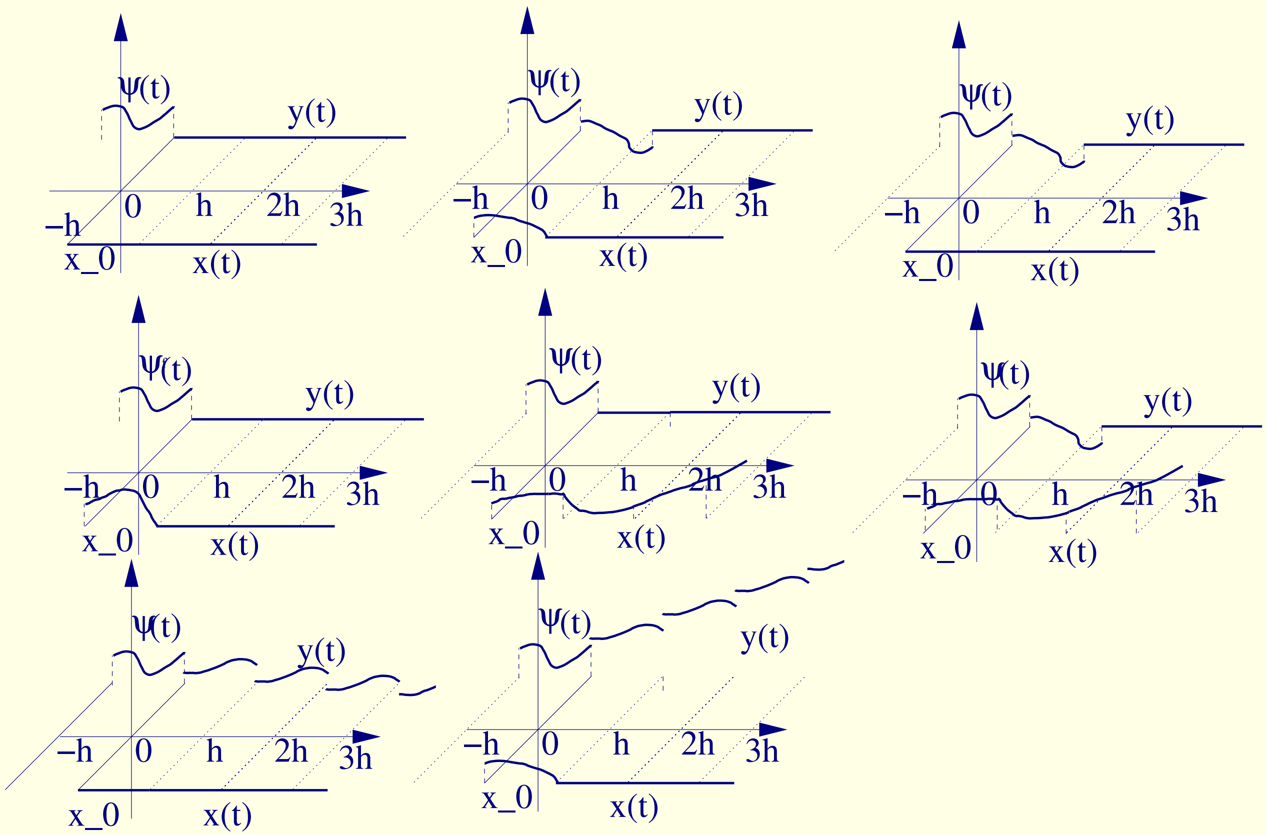

Figure 1: Small solutions of DAD systems.

Small solutions of DAD systems ...

For the theorem we need some definitions. Let $E(g)$ be the exponential type of $g : \mathbb{C} \to \mathbb{C}$ be g is an entire function of order 1 (i.e. $g(s) = O(e^{s^{1+\iota}}), \iota \geq 0$). Then

$$E(g) = \limsup_{|s| \to \infty} \frac{\log |g(se^{i\theta})|}{s}.$$

For $g : \mathbb{C} \to \mathbb{C}^q$, the exponential type is defined by

$$E(g) = \max_{1 \leq j \leq q} E(g_j).$$

Let $\Delta(p)$ denote the characteristic matrix function

$$\Delta(p) = \begin{pmatrix} pI_{n_1} - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} - A_{22}e^{-ph} \end{pmatrix}.$$

Let C_{ij} be cofactors of $\Delta(p)$. Define ε and σ by

$$E(\det \Delta(p)) = n_2 h - \varepsilon, \quad \max_{1 \leq i,j \leq n_1+n_2} E(C_{ij}(p)) = n_2 h - \sigma.$$

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Small solutions of DAD systems ...

Proposition 1. Let $x_1(\cdot)$, $x_2(\cdot)$ be solutions of (1) on $(0, \infty)$, such that at least one of the following condition holds

$$\forall k \in \mathbb{Z} \; x_1(t)e^{kt} \to 0 \text{ as } t \to +\infty, \tag{3a}$$

$$\forall k \in \mathbb{Z} \; x_2(t)e^{kt} \to 0 \text{ as } t \to +\infty. \tag{3b}$$

Then $x_1(t) = 0$ for all $t \geq \varepsilon - \sigma$ if condition (3a) holds and $x_2(t) = 0$ for all $t \geq \varepsilon - \sigma$ if condition (3b) holds.

Definition 1. We say that System (1) has *a nontrivial small solution* if there exists a solution $x_1(\cdot)$, $x_2(\cdot)$ such as conditions (3) hold and at least $x_1(\cdot)$ or $x_2(\cdot)$ is not trivial.

Definition 2. We say that System (1) has *a nontrivial small solution with respect to* x_1 [x_2] if there exists a solution $x_1(\cdot)$, $x_2(\cdot)$ such as condition (3a) [(3*b*)] holds and x_1 [x_2] is not trivial.

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Observability of nontrivial small solutions

Definition 3. The nontrivial small solutions of System (1) are said to be observable if there exists $T > 0$

$$\left.\begin{array}{r} x_1(t) \; = 0 \;\forall\; t \;\geq\; T \\ x_2(t) \; = 0 \;\forall\; t \;\geq\; T \\ z(t) \; = 0 \;\forall\; t \;>\; 0 \end{array}\right\} \Rightarrow x_1(t) = 0,\; x_2(t) = 0,\; \forall t > 0.$$

Theorem 2. *The nontrivial small solutions of System (1) are observable if and only if*

$$i)\; \max_{\lambda\in\mathbb{C}} rank \begin{bmatrix} A_{11} - \lambda I_{n_1} & A_{12} & 0 \\ A_{21} & -I_{n_2} & A_{22} \\ 0 & A_{22} & 0 \\ B_1 & B_2 & 0 \end{bmatrix} = n_1 + n_2 + rank\; A_{22}, \tag{4}$$

$$ii)\; rank \begin{bmatrix} B_2 A_{22} \\ B_2 (A_{22})^2 \\ \vdots \\ B_2 (A_{22})^{n_2} \\ (A_{22})^{n_2} \end{bmatrix} = rank \begin{bmatrix} B_2 A_{22} \\ B_2 (A_{22})^2 \\ \vdots \\ B_2 (A_{22})^{n_2} \\ (A_{22})^{n_2} \\ A_{22} \end{bmatrix}. \tag{5}$$

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Spectral observability

Definition 4. System (1) is infinite-time observable, if for all initial data for which $z(t) = 0$ for $t \in [0, \infty)$ there exists t_1 such that $x_1(t) = 0$ and $x_2(t) = 0$ for $t \in [t_1, \infty)$.

Definition 5. System (1) is spectrally observable, if all its eigenvalues are observable. An eigenvalue λ is observable if any corresponding eigensolution of the form $x_1(t) = \exp(\lambda t)x_1(0)$, $x_2(t) = \exp(\lambda t)x_2(0)$, $x_1(0) \neq 0$, $x_2(0) \neq 0$, obtains $z(t) = 0$ for $t \in [0, \infty)$.

Proposition 3. System (1) is spectrally observable if and only if

$$\operatorname{rank}\begin{pmatrix} \lambda I_{n_1} - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} - A_{22}e^{-\lambda h} \\ B_1 & B_2 \end{pmatrix} = \mathrm{n}_1 + \mathrm{n}_2, \tag{6}$$

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Proposition 4. System (1) is spectrally observable if and only if System (1) is infinite-time observable.

Weak observability of small solutions with respect to x_2

Definition 6. The nontrivial small solutions with respect to x_2 of System (1) are said to be weakly observable if

$$\exists T > 0 \quad \left.\begin{array}{rlcccc} x_1(t) & = 0 & \forall & t & \geq & 0 \\ x_2(t) & = 0 & \forall & t & \geq & T \\ z(t) & = 0 & \forall & t & \geq & 0 \end{array}\right\} \Rightarrow x_2(t) = 0, \ \forall t > 0.$$

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$$rank \begin{bmatrix} B_2 A_{22} \\ B_2 (A_{22})^2 \\ \vdots \\ B_2 (A_{22})^{n_2} \end{bmatrix} = rank \begin{bmatrix} B_2 A_{22} \\ B_2 (A_{22})^2 \\ \vdots \\ B_2 (A_{22})^{n_2} \\ A_{22} \end{bmatrix}. \quad (7)$$

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Weak observability of small solutions with respect to x_2 ...

Definition 7. We say that $x_1(t)$, $t > 0$, $x_2(t)$, $t > 0$ is a strong solution of System (1) if equations (1) are satisfied for all t, $t \geq 0$ (where by the derivative in (1a) we mean right-hand derivative for $t = 0$).

Theorem 6. *The nontrivial strong small solutions of System (1) with respect to x_2 are weakly observable if and only if*

$$\operatorname{rank}\begin{bmatrix} A_{12}A_{22} \\ \vdots \\ A_{12}A_{22}^{n_2} \\ B_2A_{22} \\ \vdots \\ B_2A_{22}^{n_2} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_{12}A_{22} \\ \vdots \\ A_{12}A_{22}^{n_2} \\ B_2A_{22} \\ \vdots \\ B_2A_{22}^{n_2} \\ A_{22} \end{bmatrix}. \tag{8}$$

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Weak observability of small solutions with respect to x_1

Definition 8. The nontrivial small solutions with respect to x_1 of System (1) are said to be weakly observable if

$$\exists T > 0 \quad \left.\begin{array}{rlllll} x_2(t) & = 0 & \forall & t & > & 0 \\ x_1(t) & = 0 & \forall & t & \geq & T \\ z(t) & = 0 & \forall & t & > & 0 \end{array}\right\} \Rightarrow x_1(t) = 0, \ \forall t > 0.$$

Theorem 8. *The nontrivial small solutions of System (1) with respect to x_1 are always weakly observable.*

Corollary 9. Spectral observability of System (1) is independent from any type of observability of small solutions of system (1)-(3).

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Example 1

Let us take System (1) in the form

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\begin{aligned}
\dot{x}_1(t) &= [1]x_1(t) + \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x_2(t),\\
x_2(t) &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_2(t-h),\\
z(t) &= [-2]x_1(t) + \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} x_2(t).
\end{aligned}
\tag{9}
$$

Computing conditions (4)

$$
\max_{\lambda\in\mathbb{C}} \operatorname{rank} \begin{bmatrix}
1-\lambda & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
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Conditions (5): $0 = \operatorname{rank}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \operatorname{rank}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1,$

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Example 1 ...

Condition (6):

$$\operatorname{rank}\begin{pmatrix} \lambda-1 & -1 & -1 & -1 \\ -1 & 1 & 0 & -e^{-\lambda h} \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -2 & 0 & 1 & 1 \end{pmatrix} \underline{\underline{\lambda=\lambda_1}} \quad 3 \neq n_1+n_2,$$

where $\lambda_1 : \ \lambda_1 - 4 - e^{-\lambda_1 h} = 0$.
Thus, system (9) is not spectrally observable and it does not observe its nontrivial small solutions nor weakly observe its nontrivial small solutions respect to x_2.
System (9) weakly observes its nontrivial strong small solutions with respect to x_2.

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Then computing conditions (4):

$$\max_{\lambda\in} \operatorname{rank} \begin{bmatrix} 1-\lambda & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} = 3 \neq n_1 + n_2 + \operatorname{rzd} A_{22},$$

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$$= 3 \neq n_1 + n_2 + \operatorname{rzd} A_{22},$$

Example 2 ...

Conditions (5): rank $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 = \text{rank} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, and (8)

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Condition (6):

$$\text{rank} \begin{pmatrix} 1-\lambda & 0 & 1 \\ 1 & -1 & e^{-\lambda h} \\ 0 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix} = 3 = n_1 + n_2.$$

Thus, system (13) is spectrally observable and it weakly observes its nontrivial small solutions with respect to x_2 but it does not observe its nontrivial small solutions.

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Thank you

for your attention